TIME, INCOME AND WEALTH DISTRIBUTION AMONG HETEROGENEOUS HOUSEHOLDS IN A TWO-SECTOR MODEL WITH SECTOR-SPECIFIC EXTERNALITIES: A SYNTHESIS OF THE ARROW-DEBREU EQUILIBRIUM THEORY AND SOLOW-UZAWA GROWTH THEORY

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Abstract:

This paper proposes a growth model of heterogeneous households with economic structure, endogenous labor supply, and sector-specific externalities. Following the economic structure of Uzawa's two sector model, we consider the economic system with one capital goods sector and one consumer goods sector. Different from the traditional Uzawa model, we consider that capital goods are also used by households as consumer durables. The model is structurally general. For instance, if the economic system has only two sectors, then the Arrow-Debreu equilibrium theory can be considered as a special case of our model. Our model is an extension of the Solow-one sector and the Uzawa two sector model. As our model also includes labor supply and durable goods (housing) as endogenous variables, it is closely related with some other growth models in the literature. The model describes a dynamic interdependence among wealth accumulation, time distribution, and division of labor under perfect competition. We find different equations which are computational for simulating the dynamics. We simulate the model, plotting the motion, identifying the equilibrium point, and confirming stability. We also examine effects of changes in the externalities, the propensity to save, the human capital, and the population on the motion of the system.

Key words: *heterogeneous households, capital goods, consumer gods, Solow-Uzawa*

2. Intoduction

Dynamics of wealth and income has always been a main concern in economic theory and empirical research. Nevertheless, the history of economic theory shows that it is not easy to properly model economic growth with wealth and income distribution. There are few formal dynamic models which explicitly deal with distribution issues among heterogeneous households in the neoclassical growth theory (Solow, 1956; Burmeister and Dobell, 1970; and Barro and Sala-i-Martin, 1995). On the other hand, the Arrow-Debreu general economic theory (see Walras, 1984; Debreu, 1959; Arrow and Hahn, 1971; and Mas-Colell *et al.*, 1995) deals with economic equilibrium issues with heterogeneous households and firms. It is desirable to integrate the economic mechanisms of the two main approaches in economics into a single analytical framework. The purpose of this study is to develop an economic model with heterogeneous households on the basis of the economic mechanisms in the Walras-Arrow-Debreu equilibrium theory and neoclassical growth theory. We introduce heterogeneous households and industries into the neoclassical growth with an alternative utility, and include endogenous time distribution and sector-externalities into the general equilibrium framework.

Relations between wealth and income distribution and growth have caused attention of economists long time ago. For instance, Kaldor (1956) argues that as income inequality is enlarged, growth should be encouraged as savings are promoted. This positive relation between income inequality and growth is also observed in studies, for instance, by Bourguignon (1981), Li and Zou (1998), Forbes (2000), and Frank (2009). There are other studies which find negative relations between income inequality and economic growth. Solow (1992) makes a hypothesis on a negative relationship between income inequality and growth. Some formal models which predicate negative relations are referred to, for instance, Galor and Zeira (1993) and Galor and Moav (2004), and Benabou (2002). Some empirical studies by, for instance, by Persson and Tabellini (1994), also confirm negative relations. This study develops a model to deal with interdependence between wealth and income distribution among heterogeneous households within the Uzawa two-sector growth modeling framework. Different households have different preferences over time distribution between leisure and work time, saving and consuming goods, housing and services. Most of extensions and generalizations of the Uzawa model are developed on the basis of a single representative household. We will generalize this type of models by introducing heterogeneous households. Another contribution of this study is to take account of consumer durables in the growth model with heterogeneous households. It should be noted that there are some models of heterogeneous households which incorporate durable goods (Cocco, 2005; Luengo-Prado, 2006; and Chambers *et al*., 2009). For instance, Dìaz and Luengo-Prado (2010) study the distribution of housing wealth and its relations with households' portfolio in a neoclassical growth framework. Their study is based on the traditional Ramsey approach with the lifetime utility (which is dependent on nondurable consumption and housing services) and is only concerned with the steady state. They also omit leisure in their model. There are few formal models which deal with economic structures and conduct genuine dynamic analysis within a single analytical framework. Our model is built on the Arrow-Debreu equilibrium theory and Solow-Uzawa growth model. The main economic mechanisms in the literature are integrated into a single framework. This study also provides some insights into demand and supply of housing markets.

This study synthesizes the ideas in the two-sector model with endogenous labor by Zhang (2005), the growth model with heterogeneous groups by Zhang (2012). The introduction of sector-specific externalities is based on, for instance, Amano and Itaya (2013). The paper is organized as follows. Section 2 introduces the basic model with wealth and income distribution among heterogeneous households with sector-specific externalities. Section 3 examines dynamic properties of the model and simulates the model with three types of households. Section 4 carries out comparative dynamic analysis with regard to the externalities, propensities to save, propensities to use leisure time, and the population. Section 5 concludes the study.

2 The basic model

The economy consists of two sectors, like in the two-sector model by Uzawa (1961). Most aspects of the production sectors are neoclassical (See Burmeister and Dobell 1970; Azariadis, 1993; and Barro and Sala-i-Martin, 1995.). Different from the Solow one-sector growth model, the Uzawa two-sector growth model treats consumption and capital goods are different commodities, which are produced in two distinct sectors. The population is constant and homogeneous. There is only one malleable capital good. In the Uzawa model, capital goods can be used as an input in both sectors in the economy. In this study, we also allow capital goods to be used by households. Capital goods are called consumer durables when they are used by households. Capital depreciates at a constant exponential rate δ_k, which is independent of the manner of use. Households own assets of the economy and distribute their incomes to consume and save. Exchanges take place in perfectly competitive markets. Factor markets work well; factors are inelastically supplied and the available factors are fully utilized at every moment. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production, labor, managerial skill and capital ownership. Each group has a fixed population, $\bar{N}_j$, ($j = 1, \ldots, J$). Let prices be measured in terms of capital goods and the price of the commodity be unity. We denote wage and interest rates by $w_j(t)$ and $r(t)$, respectively.

The total capital stock $K(t)$ is allocated between the two sectors. We use subscript index i and s to stand for capital goods and consumer goods sector, respectively. We use $N_j(t)$ and $K_j(t)$ to stand for the labor force and capital stocks employed by sector j. We use $T_j(t)$ and $\bar{T}_j(t)$ to stand for, respectively, the work time and leisure time of a typical worker in group j. The total qualified labor supply $N(t)$ is defined by

$$N(t) = \sum_{j=1}^{J} h_j T_j(t) \bar{N}_j . \quad (1)$$

We introduce

$$k_j(t) \equiv \frac{K_j(t)}{N_j(t)}, \;\; n_j(t) \equiv \frac{N_j(t)}{N(t)}, \;\; k(t) \equiv \frac{K(t)}{N(t)}, \;\; j = i, s.$$

The assumption of labor force being fully employed implies

$$N_i(t) + N_s(t) = N(t). \tag{2}$$

The capital goods sector

It is well known that in modern literature of economic growth the Cobb-Douglas production function has been widely applied to different issues (see, for instance, Lucas, 1988; Barro, 1990; and Jones, 1995). This study also uses the production function, but with externalities. We assume that production is to combine the labor force $N_i(t)$ and physical capital $K_i(t)$. The function $F_i(t)$ is specified as

$$F_i(t) = \Omega_i(t) K_i^{\alpha_i}(t) N_i^{\beta_i}(t), \;\; A_i, \alpha_i, \beta_i > 0, \;\; \alpha_i + \beta_i = 1. \tag{3}$$

where α_i and β_i are parameters. The term $\Omega_i(t)$ is sector-specific externalities in the capital goods production. This type of production functions with externalities are used by, for instance, by Benhabib and Farmer (1996), Harrison (2001), Harrison and Weder (2000), and Amano and Itaya (2013). The term of externalities is taken as given by each firm. Following Amano and Itaya (2013), we specify $\Omega_i(t)$ as

$$\Omega_i(t) = A_i K_i^{a_i}(t) N_i^{b_i}(t), \;\; A_i, a_i, b_i > 0.$$

where A_i, a_i and b_i are parameters. Markets are competitive; thus labor and capital earn their marginal products, and firms earn zero profits. The rate of interest and wage rate are determined by markets. For any individual firm $r(t)$ and $w_j(t)$ are given at each point of time. The production sector chooses the two variables $K_i(t)$ and $N_i(t)$ to maximize its profit. The marginal conditions are given by

$$r(t) + \delta_k = \alpha_i \Omega_i(t) K_i^{-\beta_i}(t) N_i^{\beta_i}(t), \;\; w_j(t) = \beta_i h_j \Omega_i(t) K_i^{\alpha_i}(t) N_i^{-\alpha_i}(t). \tag{4}$$

Consumer goods sector

We specify the production function of the education sector as follows

$$F_s(t) = \Omega_s(t) K_s^{\alpha_s}(t) N_s^{\beta_s}(t), \ \ \alpha_s, \beta_s > 0, \ \ \alpha_s + \beta_s = 1, \tag{5}$$

where the sector-specific externalities in the consumer goods production are specified as

$$\Omega_s(t) = A_s K_s^{a_s}(t) N_s^{b_s}(t), \ \ A_s, a_s, b_s > 0.$$

The marginal conditions are

$$r(t) + \delta_k = \alpha_s \, p(t) \Omega_s(t) K_s^{-\beta_s}(t) N_s^{\beta_s}(t), \ \ w_j(t) = \beta_s h_j \, p(t) \Omega_s(t) K_s^{\alpha_s}(t) N_s^{-\alpha_s}(t). \tag{6}$$

Consumer behaviors and wealth dynamics

In this study, we use an alternative approach to modeling behavior of households proposed by Zhang (1993). The preference over current and future consumption is reflected in the consumer's preference structure over leisure time, consumption and saving. Let $\bar{k}_j(t)$ stand for the per capita wealth of group j. We have $\bar{k}_j(t) = \bar{K}_j(t) / \bar{N}_j$, where $\bar{K}_j(t)$ is the total wealth held by group j. Per capita current income from the interest payment $r(t)\bar{k}_j(t)$ and the wage payment $T_j(t) w_j(t)$ is given by

$$y_j(t) = r(t)\bar{k}_j(t) + T_j(t) w_j(t).$$

We call $y_j(t)$ the current income in the sense that it comes from consumers' payment for human capital and efforts and consumers' current earnings from ownership of wealth. The sum of money that consumers are using for consuming, saving, and education are not necessarily equal to the temporary income because consumers can sell wealth to pay, for instance, the current consumption if the temporary income is not sufficient for buying food and touring the country. The total value of wealth that consumers can sell to purchase goods and to save is equal to $\bar{k}_j(t)$. Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The per capita disposable income is given by

$$\hat{y}_j(t) = y_j(t) + \bar{k}_j(t) = (1 + r(t))\bar{k}_j(t) + T_j(t) w_j(t). \tag{7}$$

The disposable income is used for saving and consumption. It should be noted that the value, $\bar{k}_j(t)$, (i.e., $p(t)\bar{k}_j(t)$ with $p(t) = 1$), in the above equation is a flow variable. Under the assumption that selling wealth can be conducted instantaneously without any transaction cost, we may consider $\bar{k}_j(t)$ as the amount of the income that the consumer obtains at time t by selling all of his wealth. Hence, at time t the consumer has the total amount of income equaling $\hat{y}_j(t)$ to distribute among saving, consumption and education. In the growth literature, for instance, in the Solow model, the saving is proportional to the current income, $y_j(t)$, while in this study the saving is chosen by maximizing the utility subject to the budget constraint. As demonstrated late on, savings is influenced by wealth as well as current income. This is different from the typical Solowian assumption on saving behavior.

The disposable income is used for saving and consumption. At each point of time, a consumer would distribute the total available budget among savings $s_j(t)$, consumer durables $k_{hj}(t)$, and consumption of goods $s_j(t)$. The budget constraint is given by

$$p(t)c_j(t)+(r(t)+\delta_k)k_{hj}(t)+s_j(t)=\hat{y}_j(t)=r(t)\bar{k}_j(t)+w_j(t)T_j(t)+\bar{k}_j(t). \qquad (8)$$

Denote $\bar{T}_j(t)$ the leisure time at time t and the (fixed) available time for work and leisure by T_0. The time constraint is expressed by

$$T_j(t)+\bar{T}_j(t)=T_0. \qquad (9)$$

Substituting (11) into (10) implies

$$w_j(t)\bar{T}_j(t)+p(t)c_j(t)+(r(t)+\delta_k)k_{hj}(t)+s_j(t)=\bar{y}_j(t), \qquad (10)$$

where

$$\bar{y}_j(t)\equiv r(t)\bar{k}_j(t)+w_j(t)T_0+\bar{k}_j(t).$$

In our model, at each point of time, consumers have four variables to decide. We assume that utility level $U_j(t)$ that the consumers obtain is dependent on the leisure time, $T_j(t)$, consumer durables, $k_{hj}(t)$, the consumption level of consumption goods $c_j(t)$, and savings $s(t)$ as follows

$$U_j(t) = \overline{T}_j^{\sigma_{0j}}(t) k_{hj}^{\eta_{0j}}(t) c_j^{\xi_{0j}}(t) s_j^{\lambda_{0j}}(t), \;\; \sigma_{0j}, \eta_{0j}, \xi_{0j}, \lambda_{0j} > 0,$$

where σ_{0j} is the propensity to use leisure time, η_{0j} is the propensity to use consumer durables, ξ_{0j} is the propensity to consume consumption goods, and λ_{0j} propensity to own wealth. It should be noted that although there are some growth models of heterogeneous households with endogenous physical capital, the heterogeneity is mostly due to differences in the initial endowments of physical capital among different types of households rather than in preferences (see, for instance, Chatterjee, 1994; Caselli and Ventura, 2000; Maliar and Maliar, 2001; and Turnovsky and Penalosa, 2008). Heterogeneous households still have essentially the same preference utility function in their approach. In our approach we consider different types of households have different utilities.

Maximizing the utility subject to (14) yields

$$w_j(t)\overline{T}_j(t) = \sigma_j \, \overline{y}_j(t), \;\; (r(t) + \delta_k) k_{hj}(t) = \eta_j \, \overline{y}_j(t), \;\; p(t) c_j(t) = \xi_j \, \overline{y}_j(t), \;\; s_j(t) = \lambda_j \, \overline{y}_j(t), \quad (11)$$

where

$$\sigma_j \equiv \rho_j \, \sigma_{0j}, \;\; \eta_j \equiv \rho_j \, \eta_{0j}, \;\; \xi_j \equiv \rho_j \, \xi_{0j}, \;\; \lambda_j \equiv \rho_j \, \lambda_{0j}, \;\; \rho_j \equiv \frac{1}{\sigma_{0j} + \eta_{0j} + \xi_{0j} + \lambda_{0j}}.$$

We now find dynamics of capital accumulation. According to the definition of $s_j(t)$, the change in the household's wealth is given by

$$\dot{\overline{k}}_j(t) = s_j(t) - \overline{k}_j(t) = \lambda_j \, \overline{y}_j(t) - \overline{k}_j(t). \quad (12)$$

This equation simply states that the change in wealth is equal to the saving minus dissaving.

Demand and supply

The output of the consumer goods sector is consumed by the households. That is

$$\sum_{j=1}^{J} c_j(t) \overline{N}_j = F_s(t). \quad (13)$$

As output of the capital goods sector is equal to the depreciation of capital stock and the net savings, we have

$$S(t) - K(t) + \delta_k K(t) = F_i(t), \tag{14}$$

where

$$S(t) \equiv \sum_{j=1}^{J} s_j(t), \quad K(t) = \sum_{j=1}^{J} \bar{k}_j(t)\bar{N}_j .$$

Capital being fully utilized

Total capital stock $K(t)$ is allocated to the two sectors and households. As full employment of labor and capital is assumed, we have

$$K_i(t) + K_s(t) + K_h(t) = K(t), \tag{15}$$

where $K_h(t)$ is consumer durables

$$K_h(t) = \sum_{j=1}^{J} k_{hj}(t)\bar{N}_j . \tag{16}$$

We competed the model. We now examine behavior of the model.

3. The dynamics and its properties

The dynamic system consists of any (finite) number of households. As behavioral patterns vary among different types, it is reasonable to expect that the dynamic system is of high dimension. The following lemma shows that the dimension of the dynamical system is equal to the number of types of households. We also provide a computational procedure for calculating all the variables at any point of time. Before stating the lemma, we introduce a new variable $z(t)$ by

$$z(t) \equiv \frac{r(t) + \delta_k}{w_j(t)/h_j}.$$

Lemma 1

The motion of the economic economy is determined by J differential equations with $z(t)$, $K_i(t)$, and $\{\bar{k}_j(t)\}$, where $\{\bar{k}_j(t)\} \equiv (\bar{k}_3(t), \cdots, \bar{k}_J(t))$, as the variables

$$\dot{z}(t) = \Lambda_1\left(z(t),\, K_i(t),\, \{\bar{k}_j(t)\}\right),$$
$$\dot{K}_i(t) = \Lambda_2\left(z(t),\, K_i(t),\, \{\bar{k}_j(t)\}\right), \quad (17)$$
$$\dot{\bar{k}}_j(t) = \Lambda_j\left(z(t),\, K(t)_i,\, \{\bar{k}_j(t)\}\right),$$

in which Λ_j are unique functions of $z(t)$, $K_i(t)$, and $\{\bar{k}_j(t)\}$ defined in Appendix. At any point of time the other variables are unique functions of $z(t)$, $K_i(t)$, and $\{\bar{k}_j(t)\}$ determined by the following procedure: $\bar{k}_1(t)$ and $\bar{k}_2(t)$ by (A15) $\rightarrow$ $r(t)$ and $w_j(t)$ by (A4) $\rightarrow$ $\bar{y}_j(t)$ by (A5) $\rightarrow$ $N(t)$ by (A10) $\rightarrow$ $K_s(t)$ by (A2) $\rightarrow$ $N_i(t)$ and $N_s(t)$ by (A1) $\rightarrow$ $F_i(t)$ by (3) $\rightarrow$ $F_s(t)$ by (5) $\rightarrow$ $p(t)$ by (6) $\rightarrow$ $\bar{T}_j(t)$, $k_{hj}(t)$, $c_j(t)$, and $s_j(t)$ by (11) $\rightarrow$ $K_h(t) = \sum_j k_{hj}(t)\bar{N}_j$ $\rightarrow$ $T_j(t) = T_0 - \bar{T}_j(t)$ $\rightarrow$ $K(t)$ by (15).

The lemma gives a computational procedure for finding out the motion of the economic system with any number of types of households. As far as economic structure and growth theory with endogenous capital are concerned, our model is general in the sense that the model is built on the basis of economic mechanisms of the Arrow-Debreu general economic theory, the Solow growth model and the Uzawa two sector model. As the nonlinear dynamic system is of high dimension, it is difficult to prove analytical properties. To study properties of the system, we calibrate the model. We specify the parameters as follows:

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 30 \\ 60 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.2 \\ 0.7 \end{pmatrix}, \begin{pmatrix} \xi_{10} \\ \xi_{20} \\ \xi_{30} \end{pmatrix} = \begin{pmatrix} 0.12 \\ 0.18 \\ 0.2 \end{pmatrix}, \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \\ \lambda_{30} \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.75 \\ 0.7 \end{pmatrix}, \begin{pmatrix} \eta_{10} \\ \eta_{20} \\ \eta_{30} \end{pmatrix} = \begin{pmatrix} 0.015 \\ 0.010 \\ 0.008 \end{pmatrix}, \begin{pmatrix} \sigma_{10} \\ \sigma_{20} \\ \sigma_{30} \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.23 \\ 0.26 \end{pmatrix},$$

$$A_i = 1.2,\; A_s = 1,\; \alpha_i = 0.32,\; \alpha_s = 0.37,\; T_0 = 1,\; \delta_k = 0.05,\; a_i = 0.07,\; b_i = 0.03,$$

$$a_s = 0.07,\; b_s = 0.03. \quad (17)$$

The population of group 3 is largest, while the population of group 1 is smallest. The human capital level of group 3 is highest, while the human capital level of group 1 is lowest. The capital goods sector and consumer goods sector's total productivities are respectively 1.2 and 1. We specify the values of the parameters, α_j, in the Cobb-Douglas productions approximately equal to 0.3 (for instance, Miles and Scott, 2005; Abel *et al*., 2007). The depreciation rate of physical capital is specified at

0.05. Group $1's$ propensity to save is 0.8 and group $3's$ propensity to save is 0.7. The value of group $2's$ propensity is between the two groups. We specify the initial conditions as follows

$$z(0) = 0.04, \quad K_i(0) = 340, \quad \bar{k}_3(0) = 3.5.$$

The motion of the variables is plotted in Figure 1. The output level of the capital goods sector falls, while the output level of the consumer goods sector rises. Both the rate of interest and price of consumer goods lower. The wage rates of all the groups are augmented slightly. Group 1 reduces work hours, group 2 changes the time distribution slightly, and group 3 augments work hours. The total labor supply falls slightly, the labor input of the capital goods sector is diminished, and the labor input of the consumer goods sector is increased. The total capital is augmented, the capital input of the capital goods sector is reduced, and both the capital input of the consumer goods sector and the consumer durables are increased. The levels of consumer goods are increased. Group $1's$ wealth and housing levels are increased, group $2's$ wealth and housing levels are diminished, and group $3's$ wealth and housing levels are slightly affected.

Figure 1 The Motion of Some Variables

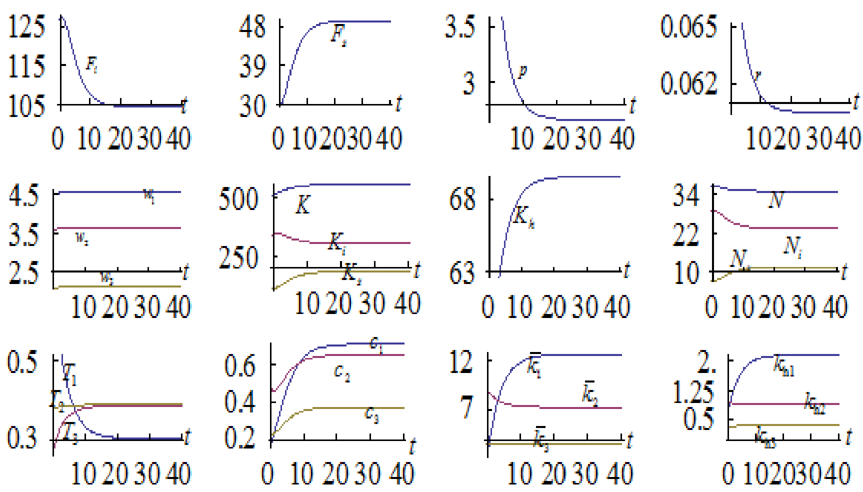


Figure 1 shows that the variables tend to become stationary. It should be noted that there are empirical studies which find negative relationships between wealth and labor supply (for instance, Holtz-Eakin et al., 1993; Cheng and French, 2000; Coronado and Perozek, 2003). In our model with the specified parameter values, the negative relationship is obvious for groups $1$and 2. Group $3's$ wealth and labor supply are slightly changed over time. The simulation confirms that the system has a unique equilibrium. We list the equilibrium values in (18). The wage rates of the three groups are respectively 4.55, 3.65, and 2.12. Group $1's$ work time is shortest, while group $3's$ work time is longest. Group $1's$ consumption, wealth and housing levels are highest, while Group $3's$ consumption, wealth and housing levels are lowest.

$$F_i = 104.35, \ F_s = 49.33, \ w_1 = 4.55, \ w_2 = 3.64, \ w_3 = 2.12, \ r = 0.06, \ p = 2.66,$$

$$N = 34.77, \ N_i = 23.39, \ N_s = 11.38, \ K = 555.18, \ K_i = 302.12, \ K_s = 183.47, \ K_h = 69.59,$$

$$\bar{k}_1 = 12.71, \ \bar{k}_2 = 7.29, \ \bar{k}_3 = 3.49, \ T_1 = 0.302, \ T_2 = 0.386, \ T_3 = 0.389, \ k_{h1} = 2.16,$$
$$k_{h2} = 0.88, \ k_{h3} = 0.36, \ c_1 = 0.72, \ c_2 = 0.66, \ c_3 = 0.38. \quad \textbf{(18)}$$

It is straightforward to calculate the three eigenvalues as follows

$$\{-0.77, \ -0.34, \ -0.26\}.$$

The eigenvalues are real and negative. The unique equilibrium is locally stable.

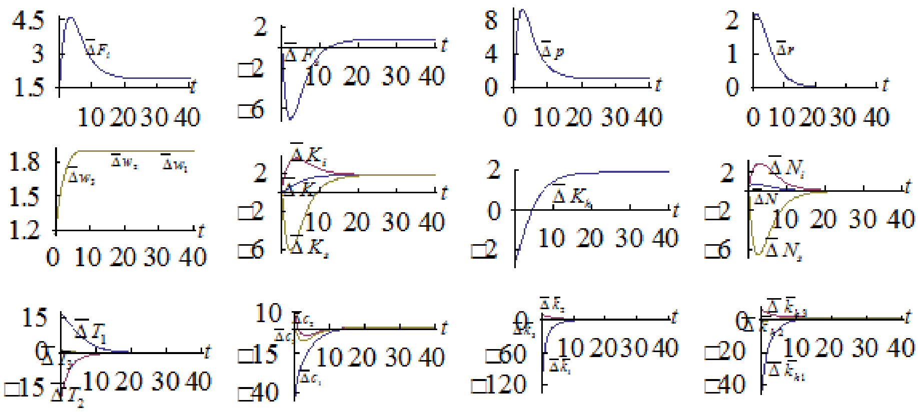
4. Comparative Dynamic Analysis

We simulated the economic system under (17). We now examine how an exogenous change in a parameter value affects the path of economic motion. As the lemma provides the procedure to follow the motion of the variables, we can easily examine effects on transitory process as well stationary state.

Capital externalities of the capital goods sector being strengthened

First, we examine the case that capital externalities of the capital goods sector is strengthened in the following way: $a_i : 0.07 \Rightarrow 0.072$. The simulation results are given in Figure 2. In the plots, a variable $\overline{\Delta} x_j(t)$ stands for the change rate of the variable, $x_j(t)$, in percentage due to changes in the parameter value. As the externalities are strengthened, the productivity of the capital goods sector is increased. The improved productivity leads to the expansion of the capital goods sector. Initially as the national capital is not increased rapidly and capital goods sector absorbs more capital, the capital input of the consumer goods sector is reduced. As the nation accumulates more capital, the capital inputs of the both sectors are enhanced. Subsequently, the output of the consumer goods sector is increased. As the capital goods sector's productivity is improved and its output increases more than the output of the consumer goods sector, the price of consumer goods is reduced. The wage rates are increased. Group $1's$ work time rises initially and subsequently falls near to its original value, group $2's$ work time falls initially and subsequently comes near to its original value, and group $3's$ work time is slightly affected. The total labor is increased slightly initially but subsequently is almost not affected. The labor distribution is initially strongly affected but is not affected in the long term. The households' time distribution, consumption and consumer durables are affected initially but subsequently only slightly affected. Hence, the long-term effects of the strengthened are strong on the macroeconomic variables and slight on the households' behavior. The rises in the wages are paid to for the increased price of consumer goods.

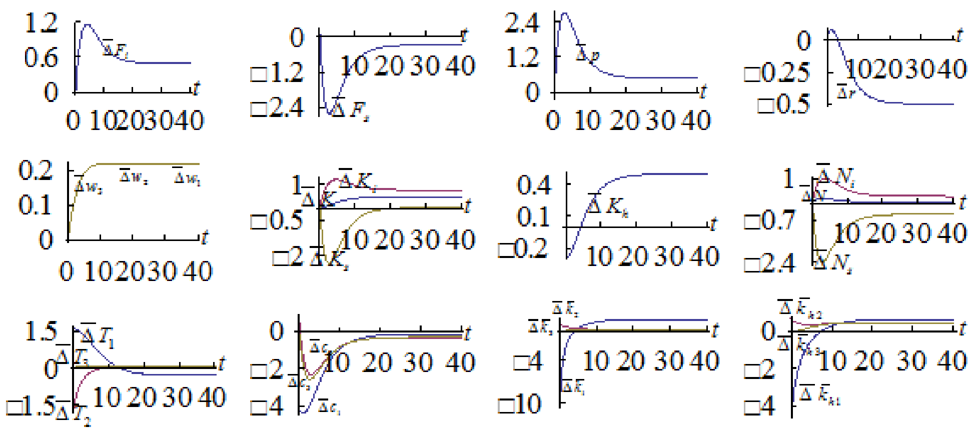
Figure 2 Capital Externalities of the Capital Goods Sector Being Strengthened



The rich group's propensity to save being augmented

We now allow the propensity to save to be changed in the following way: $\lambda_{01} : 0.8 \Rightarrow 0.81.$ The simulation results are plotted in Figure 3. As group $1's$ propensity to save is increased, the group's per capita wealth falls initially but rises subsequently. From Figure 1, we see that the group's per capita wealth is augmented rapidly in the initial period. This implies that the enhanced propensity to save slow down the wealth increase rate, even though the exogenous change has positive long term impact on the group's per capita income. The two other groups' per capita wealth levels are augmented slightly but slightly affected in the long term. The wage rates of the three groups are increased. Group $1's$ work time rises initially and subsequently falls, group $2's$ work time falls initially and subsequently comes near to its original value, and group $3's$ work time is slightly affected. The total labor is increased slightly initially but subsequently is almost not affected. More share of the labor force is re-distributed to the capital goods sector. As the nation accumulates more capital, the capital input of the capital goods sector is increased. The capital input of the consumer goods sector falls initially and subsequently approaches its original value. The consumer durables fall initially but rise subsequently. The price of consumer goods is increased as the households tend to save more from the disposable income and the output of the capital goods sector is increased. The output level of the consumer goods sector falls in association with the falling inputs. In the long term, the group which increases its propensity to save tends to benefit as its leisure, wealth, wage, and durables are increased, even though the consumption level of consumer goods is slightly reduced.

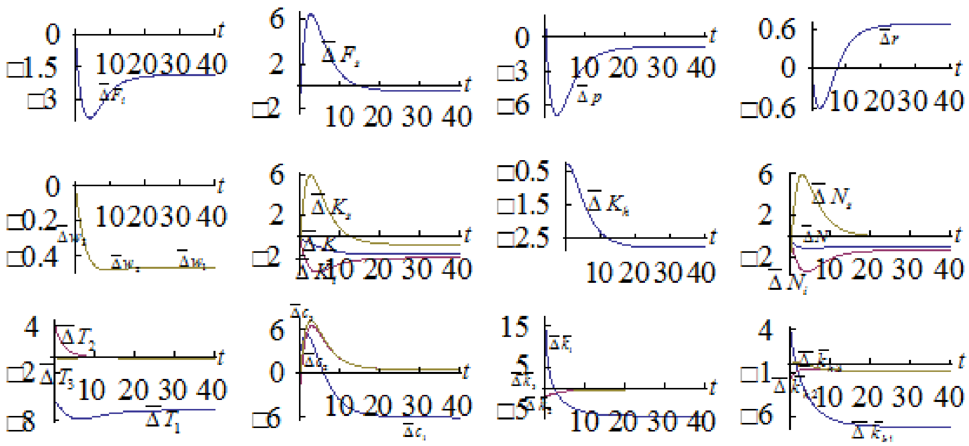
Figure 3 The Rich Group's Propensity to Save Being Augmented



The group 1's propensity to use leisure time being augmented

As a one-sector growth model of income distribution with endogenous labor supply and heterogeneous households (who are different in initial endowments of physical capital but have the same preference) within the Ramsey framework, Penalosa and Turnovsky (2006) confirm a positive equilibrium interaction between households' relative wealth and their relative allocation between work and leisure. In their model, as wealthier households have a lower marginal utility of wealth, they like to choose to increase consumption of goods and leisure. Hence, wealthier households reduce their labor supply. As the preferences are the same in this kind of the Ramsey-based models of heterogeneous agents, it is impossible to address what happen to economic dynamics when each type of household changes their preference. As people differ in their preferences, we can examine implications of change in their preferences on wealth and income distribution and economic growth. We now examine the impact of a rise in group $1's$ propensity to use leisure time in the following way: $\sigma_{01} : 0.2 \Rightarrow 0.22$. The simulation results are plotted in Figure 4. As group $1's$ propensity to use leisure time is increased, the group's work hours are reduced. The households from the group stay at home longer. The total labor supply is reduced. The consumer goods sector employs more labor initially but employs almost the same labor force as before. The labor input of the capital goods sector is reduced. The price of consumer goods falls. The rate of interest falls initially but rises subsequently. The wage rates of the three groups are reduced. The consumption level of consumer goods, wealth and durables of the household from group 1 are increased initially but decreased subsequently. The corresponding variables of the other two are slightly affected in the long term.

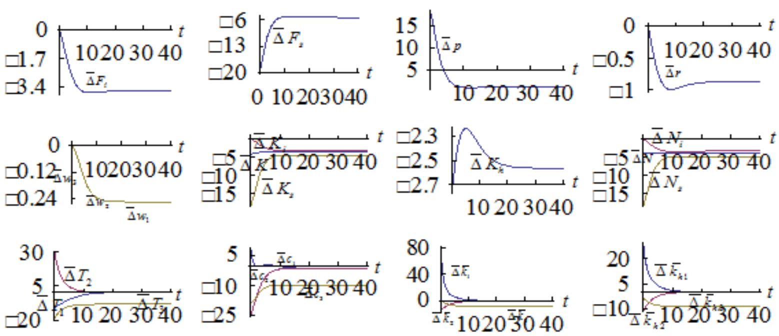
Figure 4 A Rise in Group 1's Propensity to Use Leisure Time



The group 3's propensity to use leisure time being augmented

We just examined the effects of a rise in group $1's$ propensity to use leisure time. As other groups have different preferences from group 1, it is important to compare differences in different groups' preference changes. We now study what will happen to the economic system if group $3's$ propensity to use leisure time in the following way: $\sigma_{03} : 0.26 \Rightarrow 0.3$. The simulation results are plotted in Figure 5. As group $3's$ propensity to use leisure time is increased, the group's work hours are reduced. The households from the group stay at home longer. The total labor supply is reduced. Different from the case in the change of group $1's$ preference, the consumer goods sector employs less labor. The labor input of the capital goods sector is reduced. The price of consumer goods is increased instead of being reduced as in the case of the change in group $1's$ preference. The output level of consumer goods sector is reduced. We see that there are differences in the effects of changes in the same preference variable between the high and income groups.

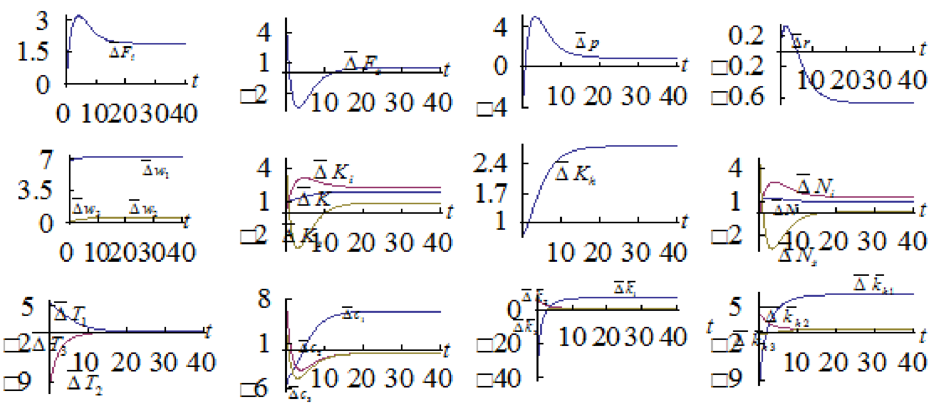
Figure 5 A Rise in Group 3's Propensity to Use Leisure Time



An augment in group 1's human capital

Group $1's$ human capital is changed as follows: $h_1 : 1.5 \Rightarrow 1.6$. We illustrate the simulation results in Figure 6. As group $1's$ human capital is enhanced, the total labor supply is augmented. Initially this increase is distributed to the two sectors. But subsequently the consumer goods sector's labor input is increased and then shifted to its original level. This implies that the increase in the labor supply is absorbed by the capital goods sector in the long term. The total capital, consumer durables and capital input to the capital goods sector are augmented. The capital input to the consumer goods sector falls initially but rises subsequently. The output level of the capital goods sector is increased. The output level of the consumer goods sector is reduced initially but slightly enhanced subsequently. The price of consumer goods falls initially but rises soon, while the rate of interest rises initially but falls soon. Group $1's$ wage rate is increased, while the other two groups' wage rates are also increased, but only slightly. Group 1 spends more time on work as the group's wage rate is increased. The wage rate approaches its original value as the group accumulates more wealth. Group $3's$ time distribution is almost not affected, while group 2 works less hours initially but works the same hours in the long term.

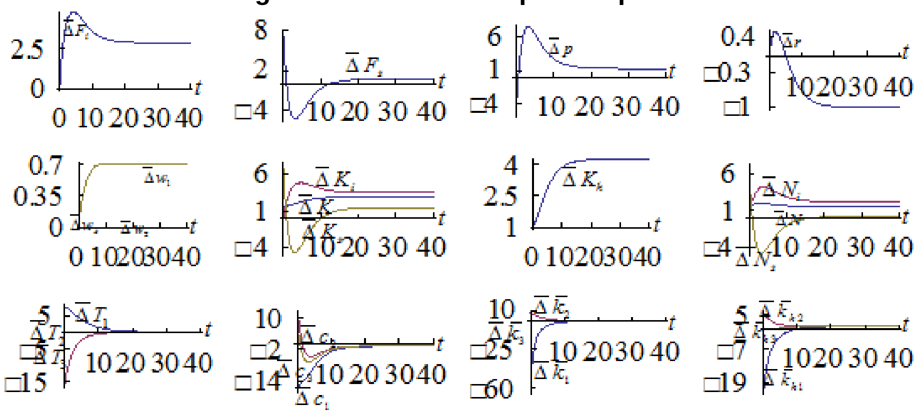
Figure 6 An Augment in Group 1's Human Capital



A rise in group 1's population

It has been pointed out that the effect of population growth varies with the level of economic development and can be positive for some developed economies. Theoretical models with human capital predict situation-dependent interactions between population and economic growth (Ehlich and Lui, 1997; Galor and Weil, 1999; and Boucekkine, *et al.,* 2002). Although our study does not consider population endogenous variables, we examine the effects of changes on economic development. We consider the following growth in group $1's$ population: $\bar{N}_1 : 10 \Rightarrow 11$. The simulation results are plotted in Figure 7. As the population is increased, the long-term per capita values of the work time, consumption level, wealth level, and consumer durables are not affected, even though these variables are affected during the motion from the old equilibrium point to the new equilibrium point. The rise in the rich group enhances the wage rates of all the groups. This occurs because the group has a highest level of human capital. The total labor force and total wealth are increased. The output level of the capital goods sector is increased. The output level of consumer goods sector rises initially, falls then, and rises in the long term.

Figure 7 A Rise in Group 1's Population



5. Concluding Remarks

This paper proposes a growth model of heterogeneous households with economic structure and sector-specific externalities. Following the economic structure of Uzawa's two sector model, we consider the economic system with one capital goods sector and one consumer goods sector. Different from the traditional Uzawa model, capital goods are also used by households as consumer durables. We also consider endogenous time distribution between work and leisure. The model describes a dynamic interdependence among wealth accumulation, time distribution, and division of labor under perfect competition. We find different equations which we can simulate to follow the motion of the economic system. We simulated the model of the economic system with three groups. We demonstrated the existence of unique equilibrium point and motion of the dynamic system. We also examined effects of changes in some parameters on the motion of the system. Our model is structurally quite general. For instance, if the economic system has only two sectors, then the Arrow-Debreu equilibrium theory (which treats capital exogenous) can be considered as a special case of our model with heterogeneous households with endogenous leisure time and wealth. It is straightforward to see that the Solow-one sector and the Uzawa two sector model are special cases of our model. As our model also includes labor supply and durable goods (housing) as endogenous variables, it is closely related with some other growth models in the literature. Because our model is structurally very general, we may generalize and extend the model in some directions. For instance, we may examine behavior of the model when the utility or production functions take on some other forms. We may extend the model to the case of multiple sectors with multiple capital goods. Our modeling framework also allows us to examine relations among entrepreneurships and wealth (see, for instance, Quadrini, 2000; Cagetti and De Nardi, 2006).

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Appendix: Proving Lemma 1

By (4) and (6), we obtain

$$z \equiv \frac{r+\delta_k}{w_j / h_j} = \frac{N_i}{\bar{\beta}_i K_i} = \frac{N_s}{\bar{\beta}_s K_s}, \tag{A1}$$

where $\bar{\beta}_j \equiv \beta_j / \alpha_j$. From (A1) and (2), we obtain

$$\bar{\beta}_i K_i + \bar{\beta}_s K_s = \frac{N}{z}. \tag{A2}$$

From (4) and the definition of Ω_i, we have

$$r = \alpha_i A_i K_i^{a_i - \beta_i} N_i^{\beta_i + b_i} - \delta_k, \ w_j = \beta_i h_j A_i K_i^{\alpha_i + a_i} N_i^{-\alpha_i + b_i}. \tag{A3}$$

Insert (A1) in (A3)

$$r = \bar{\alpha} w_0 - \delta_k, \ w_j = \bar{h}_j w, \tag{A4}$$

where

$$w_0 = K_i^a z^{\beta_i + b_i}, \ w = K_i^a z^{-\alpha_i + b_i}, \ \bar{\alpha} \equiv \alpha_i A_i \bar{\beta}_i^{\beta_i + b_i}, \ \bar{h}_j = \beta_i h_j A_i \bar{\beta}_i^{-\alpha_i + b_i}, \ a \equiv a_i + b_i.$$

From (A4) and the definitions of $\bar{y}_j$, we have

$$\bar{y}_j = \bar{w} \bar{k}_j + \bar{h}_j T_0 w, \tag{A5}$$

where $\delta \equiv 1 - \delta_k$ and $\bar{w} \equiv \bar{\alpha} w_0 + \delta$. Insert $p c_j = \xi_j \bar{y}_j$ in (13)

$$\sum_{j=1}^{J} \xi_j \, \overline{N}_j \, \overline{y}_j = p F_s .$$

Substituting (A5) in the above equation yields

$$\overline{w} \sum_{j=1}^{J} \xi_j \, \overline{N}_j \, \overline{k}_j + T_0 \, w \sum_{j=1}^{J} \xi_j \, \overline{N}_j \, \overline{h}_j = p F_s . \qquad \text{(A6)}$$

From (6), we have $p F_s = w_1 N_s / \beta_s h_1$. From this equation, (A1) and (A4), we have

$$p F_s = \frac{\overline{\beta}_s \, z \, w_1 \, K_s}{\beta_s \, h_1} . \qquad \text{(A7)}$$

Insert (A7) in (A6)

$$\overline{w} \sum_{j=1}^{J} n_j \, \overline{k}_j + w \overline{n} = \frac{\overline{\beta}_s \, z \, w_1 \, K_s}{\beta_s \, h_1} , \qquad \text{(A8)}$$

where

$$n_j \equiv \xi_j \, \overline{N}_j , \quad \overline{n} \equiv T_0 \sum_{j=1}^{J} \xi_j \, \overline{N}_j \, \overline{h}_j .$$

From (1) and (9), we have

$$N = T_0 \sum_{j=1}^{J} h_j \, \overline{N}_j - \sum_{j=1}^{J} \frac{h_j \, \sigma_j \, \overline{y}_j \, \overline{N}_j}{w_j} , \qquad \text{(A9)}$$

in which we also use $w_j \overline{T}_j = \sigma_j \, \overline{y}_j$. Substitute (A5) into (A9)

$$N = \overline{N} - \sum_{j=1}^{J} \tilde{\sigma}_j \, \overline{k}_j , \qquad \text{(A10)}$$

where

$$\overline{N} \equiv T_0 \sum_{j=1}^{J} (1-\sigma_j) h_j \overline{N}_j , \quad \tilde{\sigma}_j \equiv \frac{\overline{w} h_j \sigma_j \overline{N}_j}{w_j} .$$

From (15) and (16), we have

$$K_i + K_s + \frac{1}{r+\delta_k} \sum_{j=1}^{J} \eta_j \overline{N}_j \overline{y}_j = \sum_{j=1}^{J} \overline{k}_j \overline{N}_j , \tag{A11}$$

where we use $(r+\delta_k) k_{hj} = \eta_j \overline{y}_j$. Insert (A1) and (A2) in (A11)

$$K_i + \frac{N}{\overline{\beta}_s z} - \frac{\overline{\beta}_i K_i}{\overline{\beta}_s} + \frac{h_1}{z w_1} \sum_{j=1}^{J} \eta_j \overline{N}_j \overline{y}_j = \sum_{j=1}^{J} \overline{k}_j \overline{N}_j . \tag{A12}$$

Insert (A2) in (A8)

$$\overline{w} \sum_{j=1}^{J} n_j \overline{k}_j + w\overline{n} = \frac{z w_1}{\beta_s h_1} \left(\frac{N}{z} - \overline{\beta}_i K_i \right). \tag{A13}$$

Insert (A5) and (A10) into (A12) and (A13)

$$\phi_1 \overline{k}_1 + \phi_2 \overline{k}_2 = \phi_0 ,$$
$$\varphi_1 \overline{k}_1 + \varphi_2 \overline{k}_2 = \varphi_0 , \tag{A14}$$

where

$$\phi_j (z, K_i) \equiv \frac{\tilde{\sigma}_j \overline{N}}{\overline{\beta}_s} + z \overline{N}_j - \frac{\overline{w} h_1 \eta_j \overline{N}_j}{w_1} ,$$

$$\phi_0 \left(z, K_i, \{\overline{k}_j\}\right) \equiv \left(1 - \frac{\overline{\beta}_i}{\overline{\beta}_s} \right) z K_i + \frac{\overline{N}}{\overline{\beta}_s} - \frac{h_1 T_0 w}{w_1} \sum_{j=1}^{J} \overline{h}_j \eta_j \overline{N}_j - \sum_{j=3}^{J} \phi_j \overline{k}_j ,$$

$$\varphi_j (z, K_i) \equiv \overline{w} n_j + \frac{\tilde{\sigma}_j w_1}{\beta_s h_1} , \quad \varphi_0 \left(z, K_i, \{\overline{k}_j\}\right) \equiv \frac{w_1}{\beta_s h_1} \left(\overline{N} - \overline{\beta}_i z K_i \right) - w\overline{n} - \sum_{j=3}^{J} \varphi_j \overline{k}_j ,$$

in which $\{\overline{k}_j\} \equiv (\overline{k}_3 , \ldots \overline{k}_J)$. Solving the linear equations (A14) with $\overline{k}_1$ and $\overline{k}_2$ as the variables, we have

$$\bar{k}_j = \Omega_j\left(z, K_i, \{\bar{k}_j\}\right), \quad j = 1, 2. \tag{A15}$$

Here, we don't give the expressions of the functions in (A15) as it is straightforward and the expressions are tedious. It is straightforward to confirm that all the variables can be expressed as functions of z, K_j, and $\{\bar{k}_j\}$ by the following procedure: $\bar{k}_1$ and $\bar{k}_2$ by (A15) → r and w_j by (A4) → $\bar{y}_j$ by (A5) → N by (A10) → K_s by (A2) → N_i and N_s by (A1) → F_i by (3) → F_s by (5) → p by (6) → $\bar{T}_j$, k_{hj}, c_j, and s_j by (11) → $K_h = \sum_j k_{hj} \bar{N}_j$ → $T_j = T_0 - \bar{T}_j$ → K by (15). From this procedure (A15), and (12), we have

$$\dot{\bar{k}}_j = \bar{\Omega}_j\left(z, K_i, \{\bar{k}_j\}\right) \equiv \lambda_j \bar{y}_j - \Omega_j, \quad j = 1, 2, \tag{A16}$$

$$\dot{\bar{k}}_j = \Lambda_j\left(z, K_i, \{\bar{k}_j\}\right) \equiv \lambda_j \bar{y}_j - \bar{k}_j, \quad j = 3, ..., J. \tag{A17}$$

Taking derivatives of equation (A15) with respect to t and combining with (A17) implies

$$\dot{\bar{k}}_j = \frac{\partial \Omega_j}{\partial z} \dot{z} + \frac{\partial \Omega_j}{\partial K_i} \dot{K}_i + \sum_{j=3}^{J} \Lambda_j \frac{\partial \Omega_j}{\partial \bar{k}_j}, \quad j = 1, 2. \tag{A18}$$

Equaling the right-hand sizes of equations (A16) and (A18), we get

$$\frac{\partial \Omega_j}{\partial z} \dot{z} + \frac{\partial \Omega_j}{\partial K_i} \dot{K}_i = \bar{\Omega}_j - \sum_{j=3}^{J} \Lambda_j \frac{\partial \Omega_j}{\partial \bar{k}_j}, \quad j = 1, 2. \tag{A19}$$

Solving the linear equations (A19) with $\dot{z}$ and $\dot{K}_i$ as the variables, we have

$$\dot{z} = \Lambda_1\left(z, K_i, \{\bar{k}_j\}\right),$$
$$\dot{K}_i = \Lambda_2\left(z, K_i, \{\bar{k}_j\}\right). \tag{A20}$$

Here, we don't give the expressions of the functions in (A20) as it is straightforward and the expressions are tedious. In summary, we proved Lemma 1.